Partial Solution Set, Leon Section 5.1

5.1.1c Find the angle between $\mathbf{v} = (4, 1)^T$ and $\mathbf{w} = (3, 2)^T$.

Solution: We have $\mathbf{v}^T\mathbf{w} = 14$, $||\mathbf{v}|| = \sqrt{17}$, and $||\mathbf{w}|| = \sqrt{13}$, so the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\theta = \arccos \dfrac{14}{\sqrt{221}} \approx 0.343$ radians.

5.1.2c Using the same vectors as in the preceding problem, the vector projection of $\mathbf{v}$ onto $\mathbf{w}$ is $\dfrac{14}{13}\mathbf{w} = (42/13, 28/13)^T$. The vector projection of $\mathbf{w}$ onto $\mathbf{v}$ is $\dfrac{14}{17}\mathbf{v} = (56/17, 14/17)^T$.

5.1.3 For each pair of vectors $\mathbf{x}$ and $\mathbf{y}$, compute the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$, and verify that $\mathbf{p}$ is orthogonal to $\mathbf{x} - \mathbf{p}$.

(b) $\mathbf{x} = (3, 5)^T$, $\mathbf{y} = (1, 1)^T$

(d) $\mathbf{x} = (2, -5, 4)^T$, $\mathbf{y} = (1, 2, -1)^T$.

Solution:

(b) The vector projection is $\mathbf{p} = \dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = 4\mathbf{y} = (4, 4)^T$. Clearly $\mathbf{p}$ is orthogonal to $\mathbf{x} - \mathbf{p} = (-1, 1)^T$.

(d) The vector projection is $\mathbf{p} = \dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = -2\mathbf{y} = (-2, -4, 2)^T$. As in (b), the verification of orthogonality is trivial.

5.1.5 Find the point on the line $y = 2x$ that is closest to the point $(5, 2)$.

Solution: We want to find the vector projection of $\mathbf{v} = (5, 2)^T$ onto some vector $\mathbf{w}$ that is colinear with the given line. Any choice will do, say $\mathbf{w} = (1, 2)^T$. The projection is $\dfrac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w} = \dfrac{9}{5}\mathbf{w} = (9/5, 18/5)^T$.

5.1.7 Find the distance from the point $(1, 2)$ to the line $4x - 3y = 0$.

Solution: Let $\mathbf{x} = (1, 2)^T$. We may choose any vector $\mathbf{y}$ that is contained in the line $4x - 3y = 0$; the vector $(3, 4)^T$ will do nicely. We may proceed in two ways. Method 1: find the vector projection p of $\mathbf{x}$ onto $\mathbf{y}$, then compute the length of $\mathbf{x} - \mathbf{p}$. Method 2: find a vector $\mathbf{z}$ orthogonal to $\mathbf{y}$, and compute the scalar projection of $\mathbf{x}$ onto $\mathbf{z}$. The distance in question is the absolute value of the scalar projection. This is marginally the easier approach. The line through the origin perpendicular to $4x - 3y = 0$ is the line $3x + 4y = 0$. We can choose $\mathbf{z} = (4, -3)^T$. It follows that the scalar projection of $\mathbf{x}$ onto $\mathbf{z}$ is given by $\alpha = \dfrac{\mathbf{x}^T\mathbf{z}}{||\mathbf{z}||} = -2/5$, and the distance is $2/5 = 0.4$ units.

5.1.8 In each of the following, find the equation of the plane normal to the given vector $\mathbf{N}$ and passing through the point P_0.

(a) $\mathbf{N} = (2,4,3)^T, P_0 = (0,0,0)$.
Solution: We know that, for any point $P = (x,y,z)$ in the desired plane, the vector $\underline{PP_0}$ is orthogonal to $\mathbf{N}$. It follows that the equation of the plane is

$$\mathbf{N}^T \underline{PP_0} = 2(x - x_0) + 4(y - y_0) + 3(z - z_0) = 2x + 4y + 3z = 0.$$

(c) $\mathbf{N} = (0,0,1)^T, P_0 = (3,2,4)$.
Solution: As in (a), the desired equation is

$$\mathbf{N}^T \underline{PP_0} = 0(x - x_0) + 0(y - y_0) + 1(z - z_0) = z - 4 = 0.$$

5.1.9 Find the distance from the point $(1,1,1)$ to the plane $2x + 2y + z = 0$.

Solution: The given plane is normal to $(2,2,1)^T$ and passes through the origin. Since we want only the distance, it is the scalar projection α of $\mathbf{x} = (1,1,1)^T$ onto $\mathbf{y} = (2,2,1)^T$ that we're after. This is given by $\alpha = \dfrac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|} = \dfrac{5}{3}$.

5.1.11 If $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{y} = (y_1, y_2)^T$, and $\mathbf{z} = (z_1, z_2)^T$ are elements of $\mathbf{R}^2$, prove:

(a) $\mathbf{x}^T\mathbf{x} \geq 0$.
Solution: This follows directly from the definition: $\mathbf{x}^T\mathbf{x} = x_1^2 + x_2^2$. Since the square of a real number is nonnegative, then so must be $\mathbf{x}^T\mathbf{x}$.

(b) $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$.
Solution: This, too, follows from the definition and properties of real numbers:

$$\mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 = y_1x_1 + y_2x_2 = \mathbf{y}^T\mathbf{x}.$$

5.1.12 If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^2$, show that $||\mathbf{u} + \mathbf{v}||^2 \leq (||\mathbf{u}|| + ||\mathbf{v}||)^2$ and hence that $||\mathbf{u} + \mathbf{v}|| \leq (||\mathbf{u}|| + ||\mathbf{v}||)$. When does equality hold? Give a geometric interpretation of the inequality.

Solution: The way that this problem is stated, it is tempting to take a componentwise view of $\mathbf{u}$ and $\mathbf{v}$. Such an approach might lead to the following solution:

Assume that $\mathbf{u} = (u_1, u_2)^T$ and $\mathbf{v} = (v_1, v_2)T$. Then

$$||\mathbf{u} + \mathbf{v}||^2 = (u_1 + v_1)^2 + (u_2 + v_2)^2 = u_1^2 + v_1^2 + u_2^2 + v_2^2 + 2u_1v_1 + 2u_2v_2,$$

while

$$(||\mathbf{u}|| + ||\mathbf{v}||)^2 = \left((u_1^2 + u_2^2)^{1/2} + (v_1^2 + v_2^2)^{1/2}\right)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 + 2(u_1^2 + u_2^2)^{1/2}(v_1^2 + v_2^2)^{1/2}.$$

So it all hinges on a new inequality,

$$u_1 v_1 + u_2 v_2 \leq (u_1^2 + u_2^2)^{1/2} (v_1^2 + v_2^2)^{1/2}.$$

This might seem difficult to verify unless one notices that this can be rewritten as

$$\mathbf{u}^T \mathbf{v} \leq ||\mathbf{u}|| ||\mathbf{v}||,$$

which is precisely the Cauchy-Schwarz inequality.

But if it all hinges on Cauchy-Schwarz, is there an easier way? Yes. We can take the *matrix* point of view of our vectors $\mathbf{u}$ and $\mathbf{v}$. Here is the result:

Proof:

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}||^2 &= (\mathbf{u} + \mathbf{v})^T (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u}^T \mathbf{u} + \mathbf{v}^T \mathbf{v} + 2\mathbf{u}^T \mathbf{v} \\ &\leq \mathbf{u}^T \mathbf{u} + \mathbf{v}^T \mathbf{v} + 2||\mathbf{u}|| ||\mathbf{v}|| \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2||\mathbf{u}|| ||\mathbf{v}|| \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2, \end{aligned}$$

where the inequality in line 3 is the Cauchy-Schwarz inequality. $\square$

The second method is not limited to $\mathbf{R}^2$, but holds wherever the Cauchy-Schwarz inequality holds, and is therefore the more powerful of the two.

Regardless which approach was used, it follows that $||\mathbf{u} + \mathbf{v}|| \leq (||\mathbf{u}|| + ||\mathbf{v}||)$. Equality holds when either (a) one or both of $\mathbf{u}, \mathbf{v}$ is the zero vector, or (b) either is a scalar multiple of the other. Geometrically, this is the triangle inequality in $\mathbf{R}^2$.